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ON A MULTIAXIAL NON-LINEAR HEREDITARY CONSTITUTIVE LAW FOR NON-AGEING MATERIALS WITH FADING MEMORY†

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Abstract—A hereditary type complementary power potential is postulated which permits definition of the strain rate tensor in terms of the history and current magnitude of a certain combination of stress and stress deviator components. The strain rate tensor is decomposed into elastic and viscous portions, the latter being further separated into reversible and irreversible parts with the reversible portion being represented by Volterra type integrals. For a material obeying Norton's power law, the three-dimensional non-linear constitutive equations are derived in terms of two elastic constants and five viscous material parameters which may be determined from creep tests. These equations are applied to the derivation of general non-linear viscoelastic constitutive relations for a plate element which, in turn, are used to solve the cylindrical time-deflection problem of a long simply supported ice plate subjected to in-plane compressive forces applied along the longitudinal edges. The governing equations are solved numerically, using an incremental approach. An approximate method for the effective numerical treatment of problems involving hereditary type constitutive relations is discussed in detail.

1. INTRODUCTION

Data from uniaxial creep tests conducted at temperatures sufficiently below the melting temperature ($T \gtrsim 0.6T_m$) and at stress levels below the yield stress, σ_Y, of the material indicate that typical creep curves for common structural materials consist of three different segments, namely: (i) the primary creep stage, with decreasing viscous strain rate; (ii) the secondary creep stage with a constant strain rate; and (iii) the tertiary or accelerating creep stage during which the strain rate is increasing until failure occurs. From the point of view of the designer, the first two stages of creep may be considered to be "safe", provided the design incorporates certain measures so as to ensure that the response of all elements and portions of the structure remains within these first two stages during the operational lifetime of the structure. In many cases the primary creep stage is relatively short and can be omitted in comparison with the steady creep phase, thereby simplifying the analysis considerably. Such "steady-state" creep theories, in which the creep rate is a function only of stress level, have proven to be very useful in the treatment of even relatively complex structures[1–4]. Clearly, there are materials for which such theories are not applicable.

In this paper we present a hereditary type constitutive model which describes the primary and secondary creep stages and at the same time is tractable in numerical treatments. In applying this constitutive theory, we use ice as an example. This material is relatively close to its melting point and consequently can tolerate only a low stress level if its response is to remain within the first two stages of creep. Extensive experimental data reported, for example, in Refs [5–7], suggests that a well-defined steady creep stage along the creep curve for ice exists for stress levels below $\sigma \gtrsim -0.1T + 0.2$, where T denotes the temperature in °C and σ is expressed in MPa. Such data also shows that the tensile and compressive behaviour of this material under such relatively low stress/strain levels is similar. However, even for such "low" stress levels, the primary "hardening" creep phase is relatively long and can have a significant effect on the overall behaviour of the structure. In fact, this initial or first creep phase is usually handled on the basis of so-called "strain hardening" or "time hardening" theories which have been modelled after theories from

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plasticity[1–4], the characteristic feature of which is the fact that separate constitutive laws are postulated for loading and unloading and that any "recovery" is excluded. Some interesting variations or alternatives to these types of theories are presented in Ref. [8] where the hardening parameters are assumed to depend on maximum pre-strain. Similarly, in Ref. [9] the hardening parameters are assumed to be functions of the maximum pre-stress. As opposed to these theories, hereditary type laws are analytical in the sense that a single expression handles both positive and negative stress changes as well as recovery phenomena. In linearized form, they are widely used to define material behaviour for various plastics, as well as for low stress level response of ice, concrete and other non-metallic materials.

In this paper we limit our considerations to the first two stages of creep and discuss a non-linear hereditary constitutive law for non-ageing materials with fading memory. In particular, we postulate the existence of a complementary power potential, the elastic portion of which depends on the first and second invariants of the stress rate tensor while the viscous portion is a function only of the history and the current magnitude of the second invariant of the stress deviator tensor. Thus the usual assumption concerning incompressibility due to viscous processes, a phenomenon which is supported by a large body of experimental data, is preserved. The resulting constitutive law shows the strain rate tensor to consist of three portions:

(1) the elastic part which depends on the stress rate;

(2) the reversible viscous creep rate portion in the form of hereditary Volterra type integrals; and

(3) the irreversible viscous part being a function of the current stress level.

As an example, we particularize the general constitutive relations derived here to a material obeying Norton's power creep law. The visco-elastic properties for such materials are defined by the standard elastic constants and five additional viscous parameters, to be determined from creep test data. The feasibility of such a hypothesis, namely, that all viscoelastic parameters can be determined from creep tests in uniaxial behaviour, was shown for ice in Ref. [10].

The critical point in applying the proposed constitutive theory is in the numerical treatment of the complex derivatives of the Volterra integrals, representing the hardening rule of the viscous process. In order to make the numerical solution of these integrals tractable, an approximate method, introduced in Refs [10, 11] for the uniaxial case, and utilizing non-linear spring–dashpot elements, is generalized here for the multiaxial formulation.

Finally, the theory developed is used to analyse the time–deflection behaviour of long ice plates undergoing cylindrical bending and subjected to uniform in-plane compressive loads applied along the two longitudinal edges.

2. UNIAXIAL CONSTITUTIVE LAWS FOR NON-AGEING MATERIALS WITH FADING MEMORY

For a uniaxial stress state a general constitutive law for viscoelastic materials at a constant temperature and with memory may be written in the form[12]

$$\varepsilon = \frac{\sigma}{E} + \int_0^t \hat{F}[\sigma(\tau), t]\, \mathrm{d}\tau \tag{1}$$

where σ and ε denote stress and strain, respectively, E is Young's modulus defining the linearly elastic instantaneous response while $\hat{F}$ is a function describing the viscous non-linear properties of the material. The lower and upper limits of the integral refer to the "virgin" state of the material and the current time, respectively. A somewhat less general form of this relation, but still incorporating stress-dependent "fading memory" and ageing characteristics, is given by

$$\varepsilon = \frac{\sigma}{E} + \int_0^t \tilde{F}[\sigma(\tau), \tilde{L}(t-\tau), g(\tau)]\,d\tau. \tag{2}$$

If the functions $\tilde{L}(t-\tau)$ and $\tilde{g}(\tau)$ become stress independent, the constitutive relation becomes

$$\varepsilon = \frac{\sigma}{E} + \int_0^t F[\sigma(\tau)]\cdot L(t-\tau)\cdot g(\tau)\,d\tau \tag{3}$$

indicating that $L(t)$ and $g(t)$ are characteristic properties of the material independent of stress level.

A final restriction, by which ageing is excluded, leads to[1]

$$\varepsilon = \frac{\sigma}{E} + \int_0^t F[\sigma(\tau)]L(t-\tau)\,d\tau \tag{4}$$

a relation describing the kind of material which is to be discussed in this paper. Clearly, F must be an odd function of stress. More general forms of non-linear hereditary strain–stress relations are discussed in Ref. [12] without specific applications.

The creep rate function, $L(t)$, can be determined, at least in theory, from creep test data by substituting $\sigma = \sigma_0 = \text{const.}$ in eqn (4) to obtain

$$L(t) = \frac{1}{F(\sigma)}\cdot\frac{d\varepsilon}{dt}\bigg|_{\sigma=\sigma_0}. \tag{5}$$

For standard creep curves, exhibiting well-defined primary (hardening) and secondary (steady) creep phases, $L(t)$ may be decomposed into two parts as

$$L(t) = A_1 j(t) + A_2 \tag{6}$$

in which A_1 and A_2 are material constants associated with the first and second stages of creep, respectively, while $j(t)$ denotes a material function describing the hardening process during the primary creep phase. For convenience, one might normalize this function such that $j(0) = 1.0$, from which value it decreases with time ($dj/dt < 0$). For a well-defined steady creep stage, we further note that $j(t) \to 0$ for $t \to \infty$. Note that in writing eqn (6), the discussion, henceforth, is restricted to the first two stages of creep.

Substituting eqn (6) into eqn (4), the uniaxial strain–stress relation for this class of materials is recast in the form

$$\varepsilon = \frac{\sigma}{E} + A_1\int_0^t F[\tau(\tau)]j(t-\tau)\,d\tau + A_2\int_0^t F[\sigma(\tau)]\,d\tau \tag{7}$$

from which the strain rate, $\dot{\varepsilon}$, is obtained as

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + A_1\frac{d}{dt}\int_0^t F[\sigma(\tau)]j(t-\tau)\,d\tau + A_2 F[\sigma(t)] \tag{8}$$

a relation indicating that within the elastic response, strain rate depends on the stress rate while for the viscous portion of the response the effects of current stress and stress history are separated into two terms. As will be shown in the sequel, the second term in eqn (8) is associated with the primary (hardening) creep and is "reversible" in nature while the last term in this expression defines the "irreversible" creep resulting from the secondary (steady) creep stage.

The expression for strain rate, eqn (8), can formally be obtained from a complementary power potential, $W(\sigma)$, defined as

$$W(\sigma) = \int_0^\sigma \dot{\varepsilon}\, \mathrm{d}\sigma; \rightarrow \therefore \dot{\varepsilon} = \frac{\mathrm{d}W}{\mathrm{d}\sigma}. \tag{9}$$

For the particular class of material and the uniaxial case considered here, this potential is written in the form

$$W(\sigma) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sigma^2}{2E}\right] + A_1 \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathscr{F}[\sigma(\tau)]\cdot j(t-\tau)\,\mathrm{d}\tau + A_2 \mathscr{F}[\sigma(t)] \tag{10}$$

where

$$\mathscr{F}[\sigma] = \int F[\sigma]\,\mathrm{d}\sigma.$$

This brief review of constitutive relations for the uniaxial case was presented in order to facilitate the generalization of such relations to multiaxial states.

3. A MULTIAXIAL CONSTITUTIVE LAW FOR MATERIALS WITH FADING MEMORY

In analogy with eqn (9), postulate the existence of a complementary power potential for a three-dimensional state of stress and strain in the form

$$W(\sigma_{ij}^*) = \int_0^{\sigma_{ij}^*} \dot{\varepsilon}_{ij}\,\mathrm{d}\sigma_{ij} \tag{11}$$

in which σ_{ij}^* denotes a current state of stress and from which one immediately obtains the constitutive law in the form

$$\dot{\varepsilon}_{ij} = \frac{\partial W}{\partial \sigma_{ij}}. \tag{12}$$

Analogous complementary potentials, referred to as "creep potentials", were used in Refs [1, 9] to formulate various creep theories, some of which were mentioned in the Introduction. Potentials expressed as functions of the strain rate tensor were discussed and used in Refs [8, 13].

As for the uniaxial case (see eqn (10)), the complementary power potential, $W(\sigma_{ij})$, is assumed to consist of three parts: the first term, being related to the elastic response; the second and third terms defining reversible and irreversible creep, respectively. Invariance requirements of constitutive theory suggest that this power potential should be a function only of the three invariants of the stress tensor. As a consequence, and as a result of restricting our discussion to the linearly elastic domain, the term responsible for the elastic response is assumed to be a function of the first stress (σ_m) and second stress deviator (S) invariants, while the viscous terms are functions only of the second invariant of the stress deviator. Note that the effect of the third invariant is neglected, which is a common assumption in constitutive theory. The physical significance of these assumed functional restrictions implies material behaviour, namely incompressibility due to viscous effects, which is well supported by experimental evidence and is widely accepted in theoretical analyses. With these restrictions, the power potential for this class of materials is written in the form

$$W(\sigma_{ij}) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sigma_{\mathrm{m}}^2}{2K} + \frac{S^2}{6G}\right] + A_1\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathscr{F}[S(\tau)]\cdot j(t-\tau)\,\mathrm{d}\tau + A_2\mathscr{F}[S(t)] \tag{13}$$

in which $\mathscr{F}[S(t)]$ is the same function as that used in eqn (10), K and G denote the elastic bulk modulus and shear modulus, respectively, given by

$$K = \frac{E}{3(1-2\mu)}; \qquad G = \frac{E}{2(1+\mu)};$$

and where μ is Poisson's ratio, and σ_{m} and S are defined in the usual manner by

$$\sigma_{\mathrm{m}} = \frac{1}{3}\sigma_{kk}; \qquad S^2 = \frac{3}{2}s_{ij}s_{ji}; \qquad s_{ij} = \sigma_{ij} - \sigma_{\mathrm{m}}\delta_{ij}.$$

Using the relation given by eqn (12), we obtain the strain rate in the form

$$\dot{\varepsilon}_{ij} = \frac{1}{E}[(1+\mu)\dot{\sigma}_{ij} - 3\mu\dot{\sigma}_{\mathrm{m}}\delta_{ij}] + \frac{3}{2}A_1\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{F[S(\tau)]\cdot s_{ij}(\tau)}{S(\tau)} \cdot j(t-\tau)\,\mathrm{d}\tau + \frac{3}{2}A_2\frac{F[S(t)]}{S(t)}s_{ij}(t) \tag{14}$$

in the derivation of which use was made of the relations

$$\frac{\partial\sigma_{\mathrm{m}}}{\partial\sigma_{ij}} = \frac{1}{3}\delta_{ij}; \qquad \frac{\partial S}{\partial\sigma_{ij}} = \frac{3}{2}\frac{s_{ij}}{S}.$$

Since $s_{kk} = 0$, the rate of dilatation, $\dot{\varepsilon}_{kk}$, is written as

$$\dot{\varepsilon}_{kk} = \frac{1}{E}[(1+\mu)\dot{\sigma}_{kk} - 9\mu\dot{\sigma}_{\mathrm{m}}] = \frac{1}{K}\frac{\mathrm{d}\sigma_{\mathrm{m}}}{\mathrm{d}t} \tag{15}$$

which confirms the volume change to be "creep independent".

For the sake of notational convenience denote the viscous portion of the strain rate tensor by $\dot{\varepsilon}_{ij}^{\mathrm{v}}$, given by

$$\dot{\varepsilon}_{ij}^{\mathrm{v}} = \dot{\varepsilon}_{ij}^{\mathrm{r}} + \dot{\varepsilon}_{ij}^{\mathrm{p}} = A_1\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \hat{s}_{ij}\cdot j(t-\tau)\,\mathrm{d}\tau + A_2\hat{s}_{ij} \tag{16}$$

where

$$\hat{s}_{ij} = \frac{3}{2}s_{ij}\frac{F(S)}{S} \tag{17}$$

and where $\dot{\varepsilon}_{ij}^{\mathrm{r}}$ and $\dot{\varepsilon}_{ij}^{\mathrm{p}}$ denote the "recoverable" (reversible) and "permanent" (irreversible) portions of the viscous strain rate, respectively. For the linear case, $F(S) = S$ and eqn (14) reduces to the standard hereditary multiaxial constitutive law as given, for example, in Refs [1, 15].

In dealing with plates and shells, a plane stress state is assumed for which $\sigma_{33} = 0$ and as a result of which eqn (14) is rewritten in the form

$$\dot{\varepsilon}_{\alpha\beta} = \frac{1}{E}[(1+\mu)\dot{\sigma}_{\alpha\beta} - \mu\dot{\sigma}_{\gamma\gamma}\delta_{\alpha\beta}] + \dot{\varepsilon}^{v}_{\alpha\beta} \tag{18a}$$

$$\dot{\varepsilon}_{\alpha 3} = \frac{1+\mu}{G}\dot{\sigma}_{\alpha 3} + \dot{\varepsilon}^{v}_{\alpha 3} \tag{18b}$$

$$\dot{\varepsilon}_{33} = -\frac{\mu}{E}\dot{\sigma}_{\gamma\gamma} + \dot{\varepsilon}^{v}_{33} \tag{18c}$$

in which $\dot{\varepsilon}^{v}_{\alpha\beta}$, $\dot{\varepsilon}^{v}_{\alpha 3}$ and $\dot{\varepsilon}^{v}_{33}$ are defined by eqn (16) with $\hat{s}_{ij}$ replaced by $\hat{s}_{\alpha\beta}$, $\hat{s}_{\alpha 3}$ and $\hat{s}_{33}$, respectively, and where

$$\hat{s}_{\alpha\beta} = \frac{F(S)}{S}\left[\frac{3}{2}\sigma_{\alpha\beta} - \frac{1}{2}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right] \tag{19a}$$

$$\hat{s}_{33} = \frac{F(S)}{3S}\sigma_{\gamma\gamma}; \qquad \sigma_{\gamma\gamma} = \sigma_{11} + \sigma_{22}. \tag{19b,c}$$

The inverse relations for this plane stress case are obtained as

$$\dot{\sigma}_{\alpha\beta} = \frac{E}{(1-\mu^2)}[(1-\mu)\dot{\varepsilon}_{\alpha\beta} + \mu\dot{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta} - \dot{\bar{\varepsilon}}_{\alpha\beta}] \tag{20a}$$

in which

$$\dot{\bar{\varepsilon}}_{\alpha\beta} = A_1\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \sigma^{*}_{\alpha\beta}\cdot j(t-\tau)\,\mathrm{d}\tau + A_2\sigma^{*}_{\alpha\beta} \tag{20b}$$

and where

$$\sigma^{*}_{\alpha\beta} = \frac{F(S)}{S}\left[\frac{3}{2}(1-\mu)\sigma_{\alpha\beta} - \left(\frac{1}{2}-\mu\right)\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right] \tag{20c}$$

$$\dot{\varepsilon}_{\gamma\gamma} = \dot{\varepsilon}_{11} + \dot{\varepsilon}_{22}. \tag{20d}$$

Note that eqns (20) have effectively decomposed the total stress rate $\dot{\sigma}_{\alpha\beta}$ into an elastic, $\dot{\sigma}^{e}_{\alpha\beta}$, and a "viscous corrective" ($\dot{\sigma}^{v}_{\alpha\beta}$) term, in the form

$$\dot{\sigma}_{\alpha\beta} = \dot{\sigma}^{e}_{\alpha\beta} + \dot{\sigma}^{v}_{\alpha\beta} \tag{21a}$$

where

$$\dot{\sigma}^{e}_{\alpha\beta} = \frac{E}{(1-\mu^2)}[(1-\mu)\dot{\varepsilon}_{\alpha\beta} + \mu\dot{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta}] \tag{21b}$$

$$\dot{\sigma}^{v}_{\alpha\beta} = -\frac{E}{(1-\mu^2)}\dot{\bar{\varepsilon}}_{\alpha\beta}. \tag{21c}$$

For the case of a uniaxial stress state, say $\sigma_{11} \neq 0$, with all other $\sigma_{ij} = 0$, and $S = \sigma_{11}$, $s_{11} = \frac{2}{3}\sigma_{11}$, $s_{22} = s_{33} = -\frac{1}{3}\sigma_{11}$, the following relations are obtained from eqns (14)

$$\dot{\varepsilon}_{11} = \frac{1}{E}\dot{\sigma}_{11} + \dot{\varepsilon}^{v}_{11} \tag{22a}$$

where $\dot{\varepsilon}^{v}_{11}$ follows from eqn (16) with $\hat{s}_{ij}$ replaced by $\hat{s}_{11}$ given by

$$\hat{s}_{11} = F(\sigma_{11}). \tag{22b}$$

In directions transverse to that of σ_{11} we also obtain from eqn (14)

$$\dot{\varepsilon}_{22} = \dot{\varepsilon}_{33} = -\frac{\mu}{E}\dot{\sigma}_{11} - \frac{1}{2}\dot{\varepsilon}^{v}_{11}. \tag{23}$$

Note that relation (22a) is identical to eqn (8), expressed only in different notation. Equation (23) allows determination of the viscoelastic Poisson's ratio in the form

$$u_v(t) = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{\mu + \dfrac{E}{2\sigma_{11}(t)}\displaystyle\int_0^t \dot{\varepsilon}^{v}_{11}[\sigma_{11}(\tau)]\,\mathrm{d}\tau}{1 + \dfrac{E}{\sigma_{11}(t)}\displaystyle\int_0^t \dot{\varepsilon}^{v}_{11}[\sigma_{11}(\tau)]\,\mathrm{d}\tau}. \tag{24}$$

Since

$$\frac{1}{\sigma_{11}}\int_0^t \dot{\varepsilon}^{v}_{11}[\sigma_{11}(\tau)]\,\mathrm{d}\tau \geqq 0 \quad (\text{recall that } F(-\sigma) = -F(\sigma))$$

one can show that

$$\mu \leqq \mu_v(t) < \frac{1}{2} \qquad \text{for } 0 \leqq t < \infty \tag{25}$$

indicating that the viscoelastic process never really becomes fully incompressible.

The equations presented so far are tractable provided the reversible portion of the creep rate, represented by the derivative of a Volterra type integral, can be determined. For example, when a structure is subjected to a constant external load, the stress components may also be constant or almost constant in time. For such cases, the creep rate is written as

$$\dot{\varepsilon}^{v}_{ij} = A_1\hat{s}_{ij}\dot{j}(t) + A_2\hat{s}_{ij}. \tag{26}$$

However, for many statically indeterminate structures as well as in stability problems, despite constant external load, the stress levels vary temporally. Therefore, eqn (16) has to be used which turns out to be very cumbersome in numerical treatments. For this reason, eqn (26) is often assumed to hold even for stress states varying in time and an analogous relation is written as

$$\dot{\varepsilon}^{v}_{ij} \cong A_1\hat{s}_{ij}(t)\cdot\dot{j}(t) + A_2\hat{s}_{ij}(t). \tag{27}$$

This assumption is identical to the corresponding assumption used in the uniaxial case and referred to in the literature as Shanley's hypothesis[10, 11]. As was shown in Ref. [11] such an approximation can lead to significant discrepancies as compared with results obtained from solutions using the proper definition for creep rate, eqn (16).

In the sequel a method will be presented which will allow straight-forward treatment of the functions $\dot{\varepsilon}^{v}_{ij}$ for a material obeying Norton's power creep law.

4. MULTIAXIAL CONSTITUTIVE RELATIONS FOR MATERIAL OBEYING NORTON'S POWER LAW

The constitutive theory discussed in the previous section requires two material functions, $F(\sigma)$ and $j(t)$ to be determined experimentally. A large variety of such functions has been suggested for non-linear materials (see Refs [3, 4] for example). Here we will treat the Norton power law which seems to be applicable to a fairly wide class of materials. According to this law, the strain–stress nonlinearity is represented in the form

$$F(\sigma) = B\sigma^{n} \tag{28}$$

where B and n are material constants. Experimental data indicates that B depends strongly on temperature, while n can be assumed to be constant at least over a certain range of stress levels. Values for n range from 1.0–30[1]. For example, for ice, n has been determined to lie between 1.8 and 3.5[5, 7, 10] while for metals this number is usually much larger; for copper a value of 7.2 is suggested[9].

For a material obeying Norton's power creep law, the function $\mathscr{F}(S)$ in the complementary power potential, eqn (13), takes the specific form

$$\mathscr{F}(S) = B\frac{S^{(n+1)}}{(n+1)}. \tag{29}$$

A similar function was used in Ref. [9] for defining the hardening process the parameters of which are assumed to depend on the memory of maximum pre-stress. Introducing the new parameters $\nu_1 = 1/A_1B$ and $\nu_2 = 1/A_2B$, eqns (13) and (14) become

$$W(\sigma_{ij}) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sigma_{\mathrm{m}}^2}{2K} + \frac{S^2}{6G}\right] + \frac{1}{\nu_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{S^{n+1}}{(n+1)}j(t-\tau)\,\mathrm{d}\tau + \frac{1}{\nu_2}\frac{S^{n+1}}{(n+1)} \tag{30}$$

$$\dot{\varepsilon}_{ij} = \frac{1}{E}[(1+\mu)\dot{\sigma}_{ij} - 3\mu\dot{\sigma}_{\mathrm{m}}\delta_{ij}] + \dot{\varepsilon}^{v}_{ij} \tag{31}$$

where

$$\dot{\varepsilon}^{v}_{ij} = \frac{1}{\nu_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \tilde{\sigma}^{n}_{ij}j(t-\tau)\,\mathrm{d}\tau + \frac{1}{\nu_2}\tilde{\sigma}^{n}_{ij} \tag{32a}$$

and where

$$\tilde{\sigma}^{n}_{ij} = \frac{3}{2}s_{ij}S^{(n-1)}. \tag{32b}$$

For reasons to be shown in the sequel, $\tilde{\sigma}_{ij}$ will be referred to as the "effective viscoelastic stress".

From eqns (31) and (32), the strain rate for the uniaxial (tension) case, ($\tilde{\sigma}_{11} = \sigma_{11} = \sigma$; $\varepsilon_{11} = \varepsilon$) is written as

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{1}{v_1}\frac{d}{dt}\int_0^t \sigma^n(\tau)j(t-\tau)\,d\tau + \frac{1}{v_2}\sigma^n(t). \tag{33}$$

Equation (33) indicates that v_1 and v_2 might be looked at as some sort of "creep moduli" for the first and second stages of creep, respectively, in some sense analogous to Young's modulus for the elastic case.

4.1. *Some creep test characteristics*

The behaviour of a non-linear viscoelastic material, as defined by eqns (31) and (32), depends on the elastic constants E and μ as well as on the viscous parameters n, v_1, v_2 and the shape of the function, $j(t)$. Consequently, tests can be designed to verify the validity of our constitutive model against actual material behaviour. A first check can be obtained from tests against the characteristic features of eqn (33), which for the "creep test" at constant stress, σ_0, becomes

$$\dot{\varepsilon} = \frac{\sigma_0^n}{v_1}j(t) + \frac{\sigma_0^n}{v_2} \tag{34a}$$

from which, after integration, one obtains

$$\varepsilon(t) = \frac{\sigma_0}{E} + \frac{\sigma_0^n}{v_1}\mathscr{J}(t) + \frac{\sigma_0^n}{v_2}t \tag{34b}$$

where

$$\mathscr{J}(t) = \int_0^t j(\tau)\,d\tau. \tag{34c}$$

For various constant stress levels, the resulting strain and strain rate vs time curves should exhibit certain characteristics as indicated on Fig. 1. Firstly, let us check Norton's power law for the maximum ($t = 0$) and minimum ($t \to \infty$) strain rates. Clearly, for these two cases, eqn (34a), when plotted on a log–log scale, results in two straight lines with the slope defining n. From the same plot (Fig. 1(c)), v_1 and v_2, or alternatively β and v_2 ($\beta = v_2/v_1 + 1$) can also be determined.

The function, $j(t)$, was already normalized such that $j(0) = 1.0$. Furthermore, it was assumed that $j(t) \to 0$ as $t \to \infty$. A natural extension of this last assumption is the result $\mathscr{J}(t) \to \mathscr{J}_\infty$ for $t \to \infty$, suggesting that the cross-hatched area shown on Fig. 1(b) is limited.

If a material obeys the constitutive law given by eqn (33), the creep curves for various constant stress levels should exhibit certain invariant features based on the following reasoning: since

$$\int_0^\infty (\dot{\varepsilon} - \dot{\varepsilon}_{\min})\,dt = \frac{\sigma_0^n}{v_1}J_\infty = t_0 \tan\alpha = t_0\frac{\sigma_0^n}{v_2} \tag{35a}$$

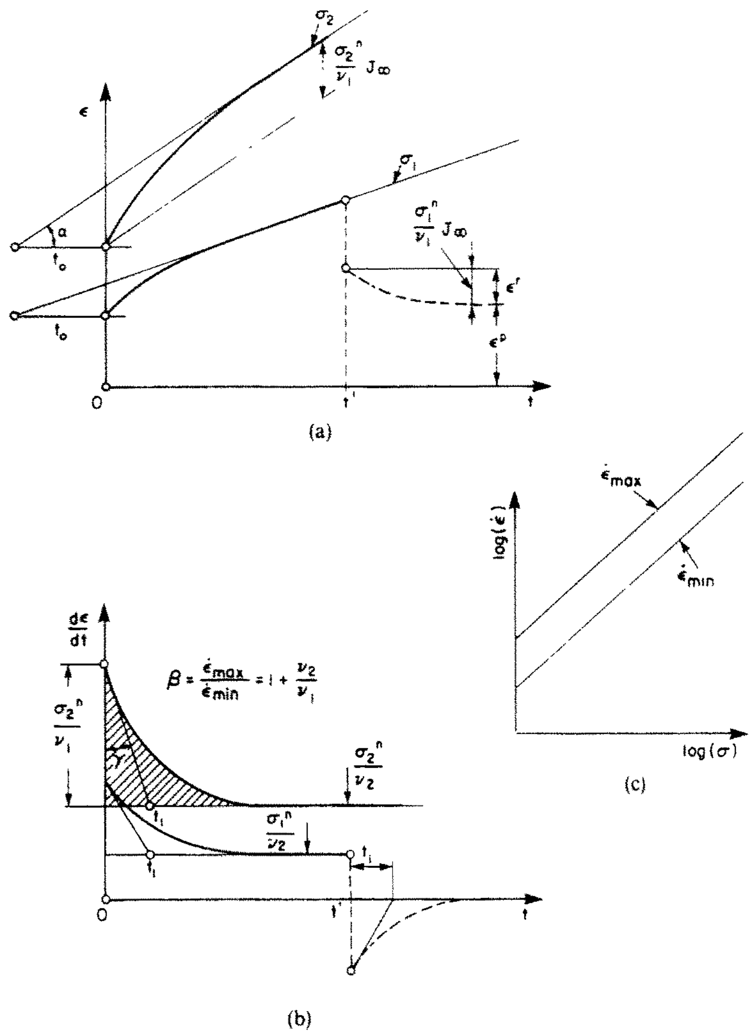

Fig. 1. Creep test characteristics of a material obeying Norton's power law. (a) Strain–time curves for two different constant stress levels. (b) Strain rate vs time curves for two different constant stress levels. (c) Maximum and minimum strain rate vs stress.

we obtain (see Fig. 1(a))

$$t_0 = \frac{v_2}{v_1}\mathscr{J}_\infty \tag{35b}$$

a parameter clearly stress independent.

From eqn (34a) we obtain

$$\left.\frac{d^2\varepsilon}{dt^2}\right|_{t=0} = \frac{\sigma_0^n}{v_1}\left.\frac{dj}{dt}\right|_{t=0} = -\cot\gamma = -\frac{\sigma_0^n}{v_1 t_1} \tag{36a}$$

from which one establishes a second stress-independent parameter, t_1, in the form (see Fig. 1(b))

$$t_1 = -\frac{1}{\left.\frac{dj}{dt}\right|_{t=0}}. \tag{36b}$$

For ice, the stress independence of t_0 and t_1 was confirmed experimentally[7] for low stress levels. These parameters were also determined in Ref. [10] for high stress levels using data given in Ref. [5]. There were significant discrepancies in these parameters as determined from various creep curves corresponding to different stress levels, discrepancies which may at least partially be due to difficulties in determining data from the creep curves resulting from the fact that stresses were so high that the curves consisted only of primary and tertiary stages.

Reversibility or irreversibility of creep strains specified by the parameters v_1 and v_2, respectively, results immediately from eqn (33). When a constant stress, σ_0, is removed at time $t = t'$, the reversible portion, amounting to $(\sigma_0^n/v_1)\mathscr{J}(t')$, should be recovered while the irreversible component, $(\sigma_0^n/v_2)t'$, remains as permanent viscous strain. If the time, t', is chosen such that the creep process is in its "steady creep" stage, measurements of the "permanent" (ε^p) and "recoverable" (ε^r) viscous strains may also be used in determining the parameter, t_0, from the relation (see Fig. 1(a))

$$t_0 = \frac{\varepsilon^r}{\varepsilon^p}t' = \frac{\varepsilon^r}{\sigma_0^n}\cdot v_2. \tag{37}$$

5. AN APPROXIMATE METHOD FOR OBTAINING THE VISCOUS PORTION OF THE STRAIN RATE

An examination of the expressions derived in the previous sections reveals that the viscous portion in those expressions are similar in the sense that they consist of a Volterra type integral part, responsible for the hereditary recoverable viscous response, and a second portion which is a function of the current stress state and is related to the permanent (irrecoverable) viscous strain. This similarity is also very apparent when we compare eqns (14) and (8), expressions for the multiaxial and uniaxial case, respectively. It is thus clear that the form of the operator is identical for the multiaxial and uniaxial case, suggesting an identical procedure for each stress component in treating this multiaxial viscous process. This, in turn, suggests approximate treatment of this phenomenon by means of one-dimensional non-linear spring–dashpot models, the suitability of which for describing such processes was demonstrated in Refs [10, 11, 14].

For a material obeying Norton's power creep law, the viscous strain rate depends on the effective viscous stress, $\tilde{\sigma}_{ij}$, as opposed to the elastic strain rate which is a function of the actual stress rate, $\dot{\sigma}_{ij}$. Thus eqn (31) can be expressed in the form

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij}(\dot{\sigma}_{ij}) + \dot{\varepsilon}^v_{ij}(\tilde{\sigma}_{ij}). \tag{38}$$

Since σ_{ij} for the uniaxial case becomes identical with $\tilde{\sigma}_{ij}$, the simple spring–dashpot models can simulate a non-linear viscoelastic process for this case while for the multiaxial formulation the elastic and viscous responses have to be modelled separately because in general, $\sigma_{ij} \neq \tilde{\sigma}_{ij}$.

As a consequence of these considerations, the tractability of these types of problems is greatly enhanced if a procedure for handling the "viscous strain rate–effective viscous stress" relation is established for the one-dimensional case. Naturally, such a procedure will have to be repeated for each non-zero $\tilde{\sigma}_{ij}$. In order to focus on this one-dimensional aspect of the procedure, let us rewrite eqn (32a) in the form

$$\dot{\varepsilon}^v(t) = \frac{1}{v_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \tilde{\sigma}^n(\tau)j(t-\tau)\,\mathrm{d}\tau + \frac{\tilde{\sigma}^n(t)}{v_2} = \dot{\varepsilon}^r(t) + \dot{\varepsilon}^p(t) \tag{39}$$

where $\dot{\varepsilon}^v$ and $\tilde{\sigma}$ denote a component of the tensors $\dot{\varepsilon}^v_{ij}$ and $\tilde{\sigma}_{ij}$, respectively. Equation (39) is identical to the analogous expression discussed in Refs [11, 14].

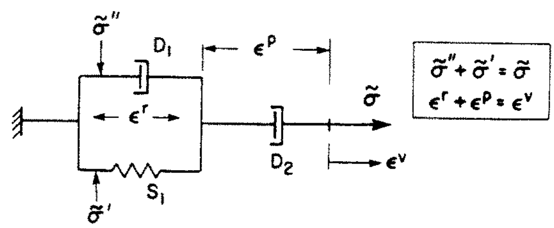

Fig. 2. Spring–dashpot model for simulation of viscous behaviour.

In considering the use of such simple models we note that the rate of permanent viscous strain can be modelled "exactly" by means of a non-linear dashpot element, D_2 (see Fig. 2), the internal constitutive relation of which reads

$$D_2 \colon \tilde{\sigma}^n = \nu_2 \frac{d\varepsilon^p}{dt}. \tag{40}$$

The reversible (recoverable) portion of the viscous strain rate, on the other hand, can be simulated only approximately by means of a non-linear Kelvin body with internal constitutive relations for its elements, D_1 and S_1, given in the form

$$D_1 \colon (\tilde{\sigma}'')^n = \nu_1 \frac{d\varepsilon^r}{dt} \tag{41a}$$

$$S_1 \colon (\tilde{\sigma}')^n + c(\tilde{\sigma})^{n-1}\tilde{\sigma}' = E_1 \varepsilon^r \tag{41b}$$

where

$$\tilde{\sigma}' + \tilde{\sigma}'' = \tilde{\sigma}; \quad \text{and} \quad \varepsilon^r + \varepsilon^p = \varepsilon^v. \tag{41c}$$

Clearly, if at any instant of time all information concerning stress and strain is available, i.e. the magnitudes $\tilde{\sigma}$, ε^v and ε^p are known, one can define the viscous strain rate for the model immediately and directly as

$$\dot{\varepsilon}^v = \frac{d\varepsilon^r}{dt} + \frac{d\varepsilon^p}{dt} = \frac{(\tilde{\sigma}'')^n}{\nu_1} + \frac{(\tilde{\sigma})^n}{\nu_2} \tag{42}$$

where the component $(\tilde{\sigma}'')$, being a part of $\tilde{\sigma}$, can be determined for a known value of ε^r from eqn (41b) using any standard iterative procedure (see Ref. [10] for details).

In eqns (41) and (42) model parameters ν_1, ν_2, E_1 and c were introduced, the first two of which are identical to the two viscous parameters used earlier in this paper. The new constants c and E_1 are to be defined in terms of the material parameters t_0 and t_1, discussed in Section 4.

We would naturally like to have our model simulate the behaviour of the material, as described by the function $j(t)$, as closely as possible. Consequently, the model shown in Fig. 2 should exhibit characteristics which are similar to those indicated for the material on Fig. 1. Let us, therefore, focus on the behaviour of the model, which obviously must have a typical creep response consisting of primary and secondary stages and must exhibit recovery.

In analogy with the creep test for the material, which allows determination of the function $j(t)$, assume the model to be subjected to a constant effective viscous stress, $\tilde{\sigma}_0$, for which the creep rate is written as

$$\dot{\varepsilon}^v|_{\tilde{\sigma}=\tilde{\sigma}_0} = \frac{\tilde{\sigma}_0^n}{\nu_1} j_m(t) + \frac{\tilde{\sigma}_0^n}{\nu_2} \tag{43}$$

where $j_m(t)$ is a characteristic function of the model describing the hardening process during the primary creep stage and where the first term defines the reversible portion of the creep rate, $\dot{\varepsilon}^r$. Comparing this part with eqn (41a), the creep rate for the corresponding element, D_1, one can write

$$\tilde{\sigma}''(t) = \tilde{\sigma}_0 \sqrt[n]{(j_m(t))} \tag{44a}$$

which when used in eqns (41b) and (41c) (for element S_1) one obtains

$$\tilde{\sigma}'(t) = \tilde{\sigma}_0(1 - \sqrt[n]{(j_m(t))} \tag{44b}$$

and

$$[1 - \sqrt[n]{(j_m(t))}]^n + c[1 - \sqrt[n]{(j_m(t))}] = \frac{E_1}{(\tilde{\sigma}_0)^n}\varepsilon^r(t). \tag{44c}$$

From this last result one can, by differentiation, obtain the reversible creep rate, which when compared with its definition (see the first term in eqn (43)) leads to

$$\frac{\mathrm{d}j_m(t)}{\mathrm{d}t} = -\frac{E_1(j_m)^{(2-1/n)}}{\nu_1\left[(1 - \sqrt[n]{j_m})^{n-1} + \dfrac{c}{n}\right]}. \tag{45}$$

This equation, together with the initial condition $j_m(0) = 1.0$, defines the model function, $j_m(t)$ for any given set of model constants, ν_1, n, c and E_1.

Before defining requirements for similarity between the model and material functions, $j_m(t)$ and $j(t)$, note that $j_m(0) = j(0) = 1.0$ and also $j_m(t) \to 0$ and $j(t) \to 0$, as $t \to \infty$. Thus the two end points of the functions are identical. Let us further define these two functions to be similar within the entire time domain if and only if

$$\left.\frac{\mathrm{d}j}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}j_m}{\mathrm{d}t}\right|_{t=0}; \qquad \text{and} \qquad \int_0^\infty j(t)\,\mathrm{d}t = \int_0^\infty j_m(t)\,\mathrm{d}t = \mathscr{J}_\infty. \tag{46a,b}$$

Substituting eqns (36b) and (45) into eqn (46a) one arrives at

$$\frac{1}{t_1} = \frac{E_1}{\nu_1}\frac{1}{(\delta + c/n)} \quad \text{where} \quad \delta = \begin{cases} 1 & \text{for } n = 1.0 \\ 0 & \text{for } n > 1.0 \end{cases}. \tag{47}$$

Equation (46b) is used, indirectly, in the following manner. From Fig. 2 we note that

$$\tilde{\sigma}''(t) \to 0; \qquad \text{as } t \to \infty \tag{48a}$$

and

$$\tilde{\sigma}'(t) \to \tilde{\sigma}_0; \qquad \text{as } t \to \infty \tag{48b}$$

which, together with eqns (43) and (41b) leads to

$$\varepsilon^r(t)|_{t\to\infty} = \frac{\tilde{\sigma}_0^n}{\nu_1}\int_0^\infty j_m(t)\,\mathrm{d}t = (1 + c)\frac{\tilde{\sigma}_0^n}{E_1}$$

from which we obtain

$$\frac{E_1}{v_1}\mathscr{J}_\infty = 1 + c. \tag{49}$$

Using eqns (47), (49) and (35b), the two additional constants, c and E_1, for the model can be related to the material parameters, t_0 and t_1 (for $n > 1.0$) by

$$c = \frac{\lambda}{1-\lambda}; \qquad E_1 = \frac{v_1}{t_1}\frac{\lambda}{n}\frac{1}{(1-\lambda)} \tag{50a,b}$$

where

$$\lambda = \frac{v_2}{v_1}\frac{t_1}{t_0}n. \tag{50c}$$

For $n = 1$, eqns (47), (49) and (35b) lead to

$$\frac{v_1}{t_1} = \frac{E_1}{1+c} = \frac{v_2}{t_0}; \qquad \lambda = 1.0. \tag{50d}$$

Thus, clearly, for the $n = 1$ case (linear material), only the ratio, $E_1/(1 + c)$ is determinable instead of E_1 and c separately. Also, $\lambda = 1.0$ and $t_0/t_1 = v_2/v_1$ for a linear material, as can easily be confirmed from a modified linear Burger's body, which our model reduces to for this special case.

The parameter λ has a simple physical meaning. Consider the "secant modulus" for the element S_1, defined as

$$\overset{s}{E}_1 = \frac{\tilde{\sigma}'}{\varepsilon_r} = \frac{E_1}{(\tilde{\sigma}')^{n-1} + c(\tilde{\sigma}_0)^{n-1}} \tag{51}$$

from which, using eqns (48b) and (50a) one obtains

$$\lambda = \frac{\overset{s}{E}_1\big|_{t\to\infty}}{\overset{s}{E}_1\big|_{t=0}}. \tag{52}$$

Since for a linear Kelvin body, $\overset{s}{E}_1 = E_1 = \text{const.}$, this relation confirms the value of λ to be unity for a linear material. Experimental data for ice shows this material parameter to be slightly less than 1.0; for example, test results presented in Ref. [7] give a value of $\lambda \cong 0.9$ while some other data[5] allows an estimate of this variable to be approximately 0.81.

Introducing the dimensionless time $\bar{t} = t/t_1$, the expression for $j_m(\bar{t})$, eqn (45), for $n > 1.0$, can be written as

$$\frac{dj_m}{d\bar{t}} = -\frac{j_m^{(2-1/n)}}{n\left(\dfrac{1}{\lambda}-1\right)(1-j_m^{1/n})^{n-1} + 1} \tag{53a}$$

which for $n = 1$ simplifies to

$$\frac{dj_m}{d\bar{t}} = -j_m; \quad \rightarrow j_m = e^{-\bar{t}}. \tag{53b}$$

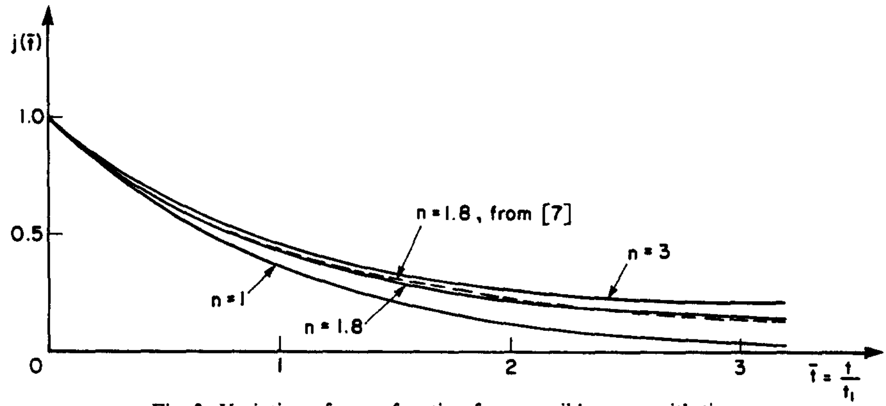

Fig. 3. Variation of creep function for reversible creep with time.

Figure 3 shows curves for $j_m(t)$ vs $\bar{t}$ for $n = 1$, 1.8, and 3, the values being obtained from a numerical solution of eqn (53a). For comparison purposes a curve based on an expression

$$j(\bar{t}) = \frac{1}{(1 + 0.5\bar{t})^2} \tag{54}$$

is also shown on the same figure, an expression which is suggested in Ref. [7] on the basis of experimental data for ice, with a value of $n = 1.8$. As can be seen from Fig. 3 the two curves for $n = 1.8$, one being experimental and the other being determined theoretically for the model, are practically identical over the entire time domain confirming the validity of our assumed "similarity" conditions.

6. EQUATIONS OF A VISCOELASTIC PLATE THEORY

Consider a plate element of thickness h subjected to stress and moment resultants, $N_{\alpha\beta}$ and $M_{\alpha\beta}$ defined by

$$N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma^z_{\alpha\beta}\, dz; \qquad M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma^z_{\alpha\beta} z\, dz \tag{55}$$

where $\sigma^z_{\alpha\beta}$ denote components of the assumed plane stress state, at a distance z from the midsurface. The stress-rate tensor for this plane stress state is given by eqns (20), using the corresponding stress and strain-rate components for the parallel surface.

Using the Kirchhoff–Love hypothesis for thin plates, which implies that $\varepsilon^z_{\alpha\beta} = \varepsilon_{\alpha\beta} + z\kappa_{\alpha\beta}$, where $\varepsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ denote midsurface (in-plane) strains and curvature changes, respectively, one obtains the stress-resultant-rate–strain-rate expressions in the form

$$\dot{N}_{\alpha\beta} = \frac{Eh}{(1-\mu^2)}[(1-\mu)\dot{\varepsilon}_{\alpha\beta} + \mu\dot{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta} - \psi'_{\alpha\beta}] \tag{56a}$$

$$\dot{M}_{\alpha\beta} = D[(1-\mu)\dot{\kappa}_{\alpha\beta} + \mu\dot{\kappa}_{\gamma\gamma}\delta_{\alpha\beta} - \psi''_{\alpha\beta}] \tag{56b}$$

where

$$\psi'_{\alpha\beta} = \frac{1}{h}\int_{-h/2}^{h/2} \dot{\bar{\varepsilon}}^z_{\alpha\beta}\, dz; \qquad \psi''_{\alpha\beta} = \frac{12}{h^3}\int_{-h/2}^{h/2} \dot{\bar{\varepsilon}}^z_{\alpha\beta} z\, dz; \qquad D = \frac{Eh^3}{12(1-\mu^2)} \tag{56c}$$

and where $\dot{\varepsilon}^z_{\alpha\beta}$ are defined by expressions analogous to eqns (20b) and (20c).

Depending on various assumptions one can make there are a number of different versions for the governing equations for such thin plates. We shall discuss here one possible version by assuming:

(a) motions of the plate elements to be such that all inertia effects are negligible;

(b) external loads cause in-plane stress resultants; the effects of which on the equilibrium of the deflected plate in the surface normal direction are significant and, consequently, the equations of equilibrium take the form

$$N_{\alpha,\beta,\alpha} + q_\beta = 0 \tag{57a}$$

$$M_{\alpha\beta,\beta\alpha} - \kappa_{\alpha\beta}N_{\alpha\beta} + q_z = 0 \tag{57b}$$

where q_α, q_z, denote surface loads per unit midsurface area;

(c) deflections to be small so as to allow the strain–displacement relations to be written as

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}); \qquad \kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{58a,b}$$

where u_α and w denote midsurface diaplacement components.

Substituting relations (58) into eqns (56) and using the result in eqns (57) one arrives at

$$\frac{Eh}{1-\mu^2}\left[\left(\frac{1+\mu}{2}\right)\dot{u}_{\alpha,\alpha\beta} + \left(\frac{1-\mu}{2}\right)\dot{u}_{\beta,\alpha\alpha} - \psi'_{\alpha\beta,\alpha}\right] + \dot{q}_\beta = 0 \tag{59a}$$

$$D[\dot{w}_{,\alpha\beta\alpha\beta} + \psi''_{\alpha\beta,\alpha\beta}] - \dot{w}_{,\alpha\beta}N_{\alpha\beta} - w_{,\alpha\beta}\cdot\frac{Eh}{1-\mu^2}\left[\left(\frac{1-\mu}{2}\right)(\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha}) + \mu\dot{u}_{\gamma,\gamma}\delta_{\alpha\beta} - \psi'_{\alpha\beta}\right] + \dot{q}_z = 0. \tag{59b}$$

These equations are linear in the displacement rates, $\dot{u}_\alpha$, $\dot{w}$. However, their solution requires knowledge of the current stress and stress history so as to allow determination of the functions $\psi'_{\alpha\beta}$ and $\psi''_{\alpha\beta}$. In addition, the current membrane stress resultants and curvature changes have to be known which demands some sort of iterative technique for the solution.

Equations (59) simplify considerably if one can assume the stress resultants and in-plane loads to be temporally invariant, i.e. $\dot{N}_{\alpha\beta} = \dot{q}_\beta = 0$. For this case eqns (59a) vanish identically, while the third equation becomes

$$D[\dot{w}_{,\alpha\beta\alpha\beta} + \psi''_{\alpha\beta,\alpha\beta}] - \dot{w}_{,\alpha\beta}N^0_{\alpha\beta} + \dot{q}_z = 0 \tag{60}$$

where $N^0_{\alpha\beta}$ denote the time-independent membrane stress resultants. The in-plane midsurface displacement rate components, $\dot{u}_\alpha$, are found from the relations

$$\dot{N}_{\alpha\beta} = 0 \rightarrow \left(\frac{1-\mu}{2}\right)(\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha}) + \mu\dot{u}_{\gamma,\gamma}\delta_{\alpha\beta} = \frac{1-\mu^2}{Eh}\psi'_{\alpha\beta}. \tag{61}$$

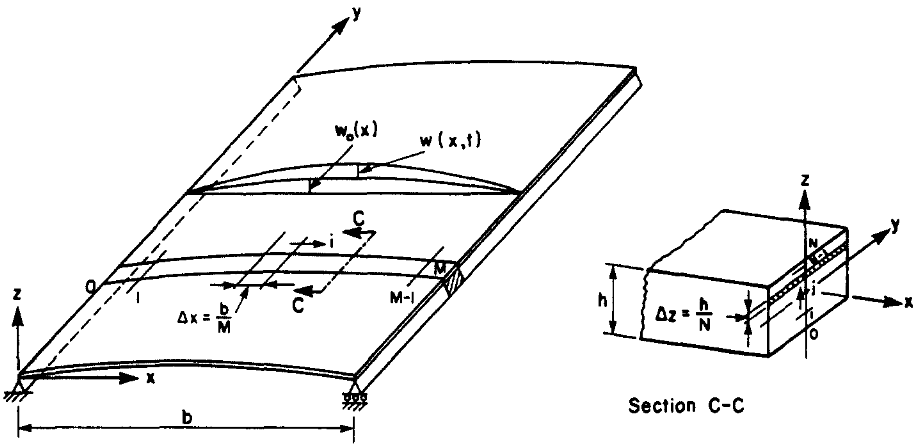

Fig. 4. Discretization of plate.

For the case of pure bending of plates, for which at $t = 0$ all membrane forces are zero, the coupling effects of the above viscoelastic plate theory still persist in the sense that even under such action there will develop, in time, midsurface in-plane displacement components.

7. CYLINDRICAL BENDING OF AN ICE PLATE

To demonstrate the effectiveness of the theory/method introduced, we treat the cylindrical bending of an imperfect long ice plate with initial displacement $w_0(x)$ and subjected to uniform in-plane compressive loads, $N_{xx} = -N_0$ applied along the longitudinal boundaries (see Fig. 4). This apparently simple case contains all characteristic features to be encountered in any viscoelastic plate problem, namely: (i) despite constant external loading, stress magnitudes vary significantly spatially and temporally; (ii) due to the non-linear constitutive law, the initial uniaxial stress state becomes two-dimensional in the course of viscous effects necessitating a two-dimensional approach throughout. The governing equation for the normal displacement, w, eqn (60) simplifies further to the form

$$D\frac{d^4\dot{w}}{dx^4} + N_0\frac{d^2\dot{w}}{dx^2} = -D\frac{d^2\psi''_{xx}}{dx^2}. \tag{62}$$

The membrane stress resultants have to satisfy the following conditions

$$\dot{N}_{xx} + \dot{N}_{xy} = 0; \qquad \int_0^b \dot{N}_{yy}\,dx = 0 \tag{63a,b}$$

where b denotes the plate width and where the second of these conditions shows that the plate is free to expand in the longitudinal (y) direction.

Due to "cylindrical bending", $\dot{\varepsilon}_{yy}$ is independent of x. Using eqns (56), one obtains the following relations for the determination of the in-plane strain rate components

$$\dot{\varepsilon}_{xx} + \mu\dot{\varepsilon}_{yy} = \psi'_{xx}; \qquad \dot{\varepsilon}_{yy} + \frac{\mu}{b}\int_0^b \dot{\varepsilon}_{xx}\,dx = \frac{1}{b}\int_0^b \psi'_{yy}\,dx. \tag{64a,b}$$

Having determined $\dot{\varepsilon}_{xx}$ and $\dot{\varepsilon}_{yy}$ (clearly $\dot{\varepsilon}_{xy} = 0$ due to symmetry), one can find $\dot{N}_{yy}$ using eqn (56). Finally $\dot{u}_x = \dot{u}$ and $\dot{u}_y = \dot{v}$ can be established from eqns (58).

Note that for $t = 0$, $\psi'_{yy} = 0$ and the elastic response can be treated as a uniaxial one. However, for $t > 0$, $\psi'_{yy} \neq 0$ one must take into account the two-dimensional stress state and the coupling effect inherent in eqns (62) and (64).

The form of all equations derived suggests an incremental procedure in which calculations are performed in temporal steps $\Delta\bar{t} = \Delta t/t_p$, where $t_p = \nu_1/\sigma_0^n$, $\sigma_0 = N_0/h$. Such "average stress"-dependent scaling of the time proved to be very convenient in numerical calculations for the uniaxial case[10] since it permits automatic "extension" or "shrinkage" of the real time steps, Δt, in terms of the viscous properties of the material and the average loading on the plate.

Spatially the plate is divided into M and N sectors along the x (width) and z (thickness) directions, respectively (see Fig. 4). Magnitudes of all required functions are calculated for each nodal point with coordinates x_i and z_j. Integration through the thickness is performed using Simpson's rule. In the numerical analysis the following procedure was used.

(1) At each nodal point the value of the total stress, total strain and the recoverable (reversible) and permanent (irreversible) strain are taken from the previous time step and stored. For $t = 0$ the elastic solution is taken.

(2) Using the equations presented, the rates for displacements, strains and stresses, at this time, are determined at every nodal point.

(3) Assuming some time interval, Δt, the increments for all variables during this time interval are calculated.

(4) The state of deformation and stress, for the time $(t + \Delta t)$ is determined at every node point of the plate by adding the increments to the previous values.

(5) Steps (1)–(4) are repeated.

This general procedure is discussed in detail in Ref. [16].

Calculations were performed for an ice plate using viscoelastic material properties taken from Ref. [7] and used also in Ref. [14] in the analysis of an imperfect ice column. The tests reported in Ref. [7] were carried out under relatively low stress levels so as to permit the development of a well-defined steady creep stage.

For a temperature of $-5°C$, the following material properties were used:

$$E = 4.25\,\mathrm{GPa};\ \mu = 0.33$$
$$n = 1.8;\ \nu_1 = 2.159 \times 10^{13}\,(\mathrm{Pa})^n\,\mathrm{h};\ \nu_2 = 1.079 \times 10^{15}\,(\mathrm{Pa})^n\,\mathrm{h}$$
$$t_0 = 100\,\mathrm{h};\ t_1 = 1.0\,\mathrm{h}.$$

The geometry of the plate was assumed to be

$$h = 3.5\,\mathrm{cm};\ b = 100\,\mathrm{cm}$$

with an initial (imperfection) displacement $0.01\,\mathrm{mm} \leqq w_0 \leqq 10\,\mathrm{mm}$. The plate is subjected to a uniform in-plane "membrane" force, $N_0 = 8750\,\mathrm{N\,m^{-1}}$ resulting in an average membrane stress $\sigma_0 = 0.25\,\mathrm{MPa}$, which is low enough to result in a well-defined steady creep stage. This stress level, σ_0, also leads to a value $t_p = 4.149 \times 10^3\,\mathrm{h}$. The plate is divided into 10 sectors in the x- and z-directions, respectively, thus creating 100 nodal points for every one of which the above described numerical procedure must be carried out. As an output of such calculations, the complete stress, strain and deflection histories for each node point are obtained.

Figure 5(a) shows the time–deflection behaviour at the centre of the plate (line along the midspan) for various initial imperfection magnitudes. As was the case for the imperfect column, the deflections increase monotonically in time with the rate accelerating as the deflection magnitude approaches a value approximately equal to the plate thickness. For comparison purposes, the time–deflection behaviour of an ice column subjected to the same compressive stresses and initial imperfections and taken from Ref. [11] is also shown on the figure. As is to be expected, the plate is slightly stiffer than the corresponding column, leading to increasing discrepancies in the time–deflection behaviour of the two structures as time passes.

The stress distribution in the plate for $\bar{t} \cong 0.056$ ($t \cong 9.7$ days) is shown in Fig. 5(b)

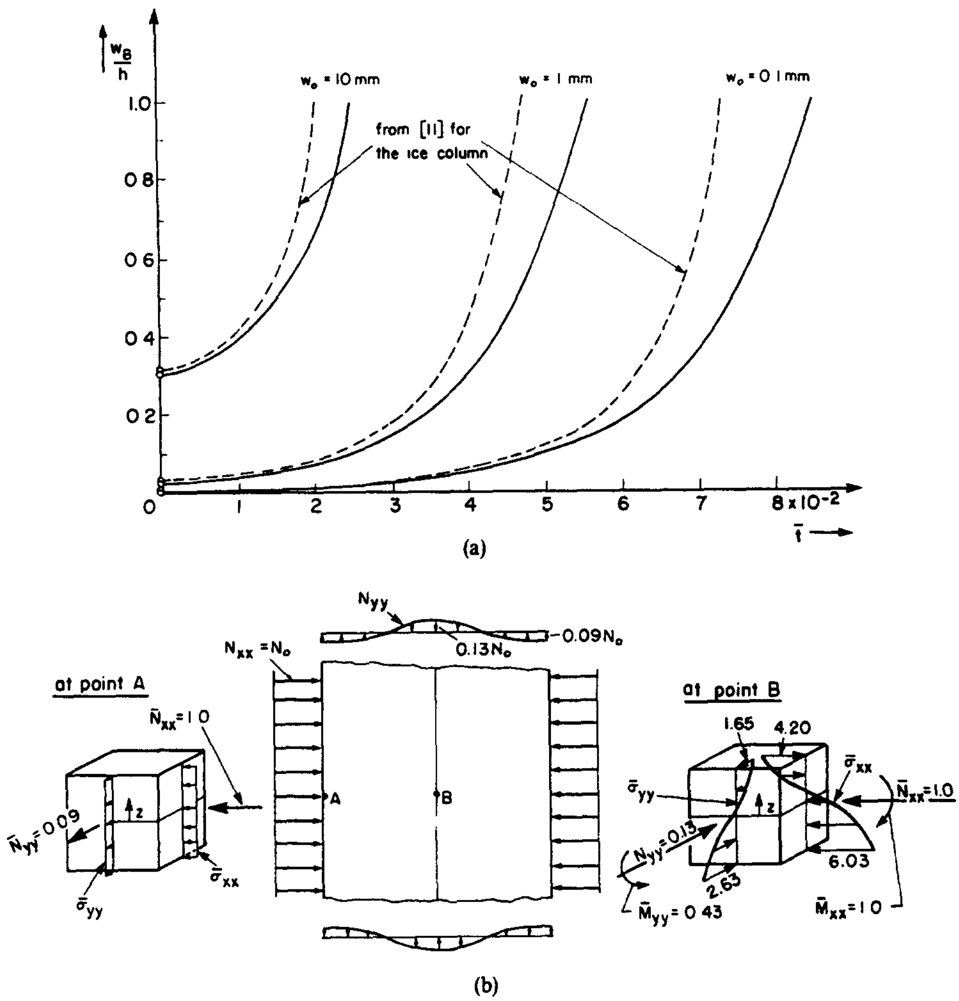

Fig. 5. Viscous behaviour of imperfect ice plate undergoing cylindrical bending. (a) Time–deflection behaviour. (b) Stress resultant and stress distribution for $\bar{t} = 0.056$ (w_0 = 1 mm, $w_B = h$).

when the total mid-span deflection, w_B, has grown to $w_B = h = 3.5$ cm. Note that by this time the "coupling effect" in the equations has resulted in some (nonzero) N_{yy} and that the stress variation across the plate thickness has become quite nonlinear.

8. CONCLUSIONS

The existence of a complementary power potential is postulated, which allows definition of a hereditary multiaxial constitutive law for non-ageing materials with fading memory. The general equations reduce to the corresponding multiaxial linear constitutive law for hereditary materials involving Volterra type integrals[1, 15], equations which are used to describe the time-dependent behaviour of non-metallic materials such as plastics. The general equations derived also reduce to the linear and non-linear uniaxial constitutive relations which have been proposed and used by many authors[1, 13].

The theory and results derived here are particularized for a material obeying Norton's power creep law. The resulting equations, when reduced to the one-dimensional case, again are in agreement with corresponding equations introduced and used by others heretofore[1–4]. What is also significant in the present theory is that the non-linear time-dependent material behaviour is specified by means of two elastic and five viscous material parameters which have clearly definable physical meaning facilitating their experimental determination. The difficulties associated with evaluating the complex derivatives of the Volterra type integrals in such constitutive relations are overcome by introducing an approximate procedure which involves the use of simple arrangements of non-linear spring–dashpot elements.

The constitutive theory derived was used to analyse the time–deflection behaviour of a long imperfect ice plate. The results from this analysis show great similarity with corresponding results for an imperfect ice column, discussed in Ref. [11]. As one would expect, the ice plate, due to the two-dimensionality of the problem, exhibits slightly larger stiffness as compared to the ice column.

Throughout the paper small strain approximation was used and the level of stress was restricted to be below the yield stress of the material. The theory developed here treats only the first two stages of creep and does not simulate the strain softening phenomenon exhibited by various materials at high stress levels and large strains. Despite this limitation, the constitutive model and associated solution technique presented here seems to provide an improved and effective tool for the analysis of viscoelastic structures, including hereditary effects.

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